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Twist-deformed supersymmetries in non-anticommutative superspaces

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Abstract

We consider a quantum group interpretation of the non-anticommutative deformations in Euclidean supersymmetric theories. Twist deformations in the corresponding superspaces and supergroups are constructed in terms of the left supersymmetry generators. Non-anticommutative $\star$-products of superfields are covariant objects in the twist-deformed supersymmetries, and this covariance guarantees the manifest invariance of superfield actions using $\star$-products.
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1. Introduction

The most popular classical non-commutative field theory (see, e.g., review [1]) can be realized on ordinary smooth field functions $f(x), g(x)$ on R^4 using the following pseudolocal representation of the $\star$-product:

$$f \star g = f e^P g = fg + \frac{i}{2}\vartheta_{mn}\partial_m f \partial_n g - \frac{1}{8}\vartheta_{mn}\vartheta_{rs}\partial_m\partial_r f \partial_n\partial_s g + \cdots,$$
$$f P g = \frac{i}{2}\vartheta_{mn}\partial_m f \partial_n g, \tag{1.1}$$

where x_m are the coordinates of R^4, $\partial_m = \partial/\partial x_m$, and ϑ_{mn} are some constants ($m, n = 1, 2, 3, 4$). All products of the functions and their derivatives in the right-hand side are commutative. It is evident that non-linear inter-

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actions in these non-commutative (non-local) field theories are not invariant with respect to the standard Lorentz transformations of local fields.

The quantum group structures in this non-commutative algebra of functions were found and analyzed in [2–5]. The basic point of this interpretation is connected with the twist operator acting on tensor products of functions

$$\mathcal{F}=\exp(\mathcal{P}),\quad \mathcal{P}=\frac{i}{2}\vartheta^{mn}P_m\otimes P_n \tag{1.2}$$

where $P_m f=\partial_m f$. The strict definition of the non-commutative product is

$$f\star g=\mu\circ\mathcal{F}f\otimes g,\quad \mu\circ f\otimes g=fg, \tag{1.3}$$

where μ is the multiplication map in the commutative algebra. Thus, this twist operator is the quantum-group analog of the pseudolocal operator $\exp(P)$ (1.1).

Let us consider generators of the Poincaré group P_m and M_{mn}. By definition, the twist-deformed Poincaré group $U_t(P_m,M_{mn})$ has the undeformed Lie algebra of generators; however, its coproduct is deformed

$$\Delta_t(P_m)=P_m\otimes 1+1\otimes P_m,\qquad \Delta_t(M_{mn})=\exp(-\mathcal{P})(M_{mn}\otimes 1+1\otimes M_{mn})\exp(\mathcal{P}). \tag{1.4}$$

The exact constructions of maps between differential operators on commutative and non-commutative algebras of functions were formulated in recent papers of the Munchen group [4]. It was shown that the $\star$-product (1.3) transforms covariantly in $U_t(P_m,M_{mn})$, but the Leibniz rule for deformed transformations is changed according to Eq. (1.4). 4D-space integrals of the covariant $\star$-products of fields are invariant with respect to $U_t(P_m,M_{mn})$. The quadratic free interactions possess also the standard Poincaré invariance.

We shall consider the quantum group interpretation of the non-anticommutative deformations in the Euclidean supersymmetric theories [6–8].[1] The basic $\star$-product of these models is realized on the standard local superfields, and supersymmetry generators can be presented as the 1st-order differential operators on the undeformed superspace. The left-handed Grassmann coordinates of these superspaces do not anticommute with respect to the $\star$-product, but the basic chiral bosonic coordinates commute with all superspace coordinates. The non-anticommutative superspace is defined exactly as the $\star$-product algebra on ordinary functions of the superspace coordinates. The twist elements for the nilpotent deformations can be constructed in terms of the left supersymmetry generators.

Section 2 is devoted to the analysis of the twist deformation of the $N=(\frac{1}{2},\frac{1}{2})$ supersymmetry [10]. We derive the unusual Leibniz rules for the deformed transformations on the products of superfields or the products of component fields. Twist deformation of the Euclidean $N=(1,1)$ supersymmetry in the chiral and harmonic superspaces is considered in Section 3. The corresponding nilpotent operator $\mathcal{P}$ is analogous to the basic bi-differential operator of the $N=(1,1)$ deformations in Refs. [7,8].

The twist interpretation allow us to understand correctly transformation properties of $\star$-products of superfields by analogy with the deformed transformations of the $\star$-products of fields (1.1) in the non-commutative field theory [4]. At the level of the pseudolocal superfield formalism, the t-supersymmetry is equivalent to the *$\star$-covariance principle* for the non-commutative algebra of superfields which means the similarity of transformations of local superfields and their $\star$-products. The covariance principle allow us to obtain a simplified field-theoretical derivation of the unusual Leibniz rules for the deformed supersymmetry transformations on $\star$-products of superfields. Known deformed supersymmetric actions in the non-anticommutative superspaces are manifestly invariant with respect to the corresponding twist-deformed supersymmetry, and this invariance explains naturally all selection rules of these theories which could seem formal earlier.

[1] Note that the alternative quantum-group deformations of supersymmetries with a more complex supersymmetric geometry were considered earlier [9], however, we shall not discuss these models here.

2. Twist-deformed $N=(\frac{1}{2},\frac{1}{2})$ supersymmetry

We use the chiral coordinates $z^M=(y_m,\theta^\alpha,\bar\theta^{\dot\alpha})$ in the Euclidean superspace R(4|2, 2), where $m=1,2,3,4$, $\alpha=1,2$, and $\dot\alpha=\dot1,\dot2$. The central and antichiral 4D coordinates are, respectively,

$$x_m=y_m-i\theta\sigma_m\bar\theta,\qquad \bar y_m=y_m-2i\theta\sigma_m\bar\theta \tag{2.1}$$

and $(\sigma_m)_{\alpha\dot\alpha}$ are the SO(4) Weyl matrices. Note that these coordinates are pseudoreal with respect to the special conjugation [7]

$$(y_m)^*=y_m,\qquad \left(\theta^\alpha\right)^*=\varepsilon_{\alpha\beta}\theta^\beta,\qquad \left(\bar\theta^{\dot\alpha}\right)^*=\varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}, \tag{2.2}$$

so one can use the reality condition for the even Euclidean chiral superfield $\phi(y,\theta)$. The generators of the Euclidean $N=(\frac{1}{2},\frac{1}{2})$ supersymmetry SUSY$(\frac{1}{2},\frac{1}{2})$ have the following form:

$$\begin{aligned}
&L^\beta_\alpha=L^\beta_\alpha(y)+L^\beta_\alpha(\theta)=\frac{1}{4}(\sigma_m\bar\sigma_n)^\beta_\alpha(y_n\partial_m-y_m\partial_n)+\theta^\beta\partial_\alpha-\frac{1}{2}\delta^\beta_\alpha\theta^\gamma\partial_\gamma,\\
&R^{\dot\beta}_{\dot\alpha}=R^{\dot\beta}_{\dot\alpha}(y)+R^{\dot\beta}_{\dot\alpha}(\bar\theta)=\frac{1}{4}(\bar\sigma_m\sigma_n)^{\dot\beta}_{\dot\alpha}(y_m\partial_n-y_n\partial_m)+\bar\theta^{\dot\beta}\bar\partial_{\dot\alpha}-\frac{1}{2}\delta^{\dot\beta}_{\dot\alpha}\bar\theta^{\dot\gamma}\bar\partial_{\dot\gamma},\\
&O=\theta^\alpha\partial_\alpha-\bar\theta^{\dot\alpha}\bar\partial_{\dot\alpha},\qquad Q_\alpha=\partial_\alpha,\qquad \bar Q_{\dot\alpha}=\bar\partial_{\dot\alpha}-2i\theta^\alpha\partial_{\alpha\dot\alpha},\qquad P_m=\partial_m,
\end{aligned} \tag{2.3}$$

where $(\bar\sigma_m)^{\dot\alpha\alpha}=\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}(\sigma_m)_{\beta\dot\beta}$, and $\partial_M=(\partial_m,\partial_\alpha,\bar\partial_{\dot\alpha})$ are partial derivatives in the chiral coordinates. Generators L^β_α, $R^{\dot\beta}_{\dot\alpha}$ and O correspond to the automorphism group $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{O}(1,1)$. The SUSY$(\frac{1}{2},\frac{1}{2})$ transformations can be separated as follows

$$\begin{aligned}
&\delta A=-(g+G)A,\\
&\quad g=P_c+R_\rho+Q_\epsilon,\quad G=L_\lambda+aO+\bar Q_{\bar\epsilon},\quad P_c=c_mP_m,\quad L_\lambda=\lambda^\alpha_\beta L^\beta_\alpha,\quad R_\rho=\rho^{\dot\alpha}_{\dot\beta}R^{\dot\beta}_{\dot\alpha},\\
&\quad Q_\epsilon=\epsilon^\alpha Q_\alpha,\quad \bar Q_{\bar\epsilon}=\bar\epsilon^{\dot\alpha}\bar Q_{\dot\alpha},
\end{aligned} \tag{2.4}$$

where the corresponding combinations of operators and transformation parameters are introduced. These definitions will be convenient in the deformed supersymmetry.

We shall use the notation S(4|2, 2) or C(4|2, 0) for the supercommutative algebras of general or chiral superfields. The bilinear multiplication map μ connects the tensor product of superfields with the local supercommutative product in S(4|2, 2)

$$\mu\circ A\otimes B=AB=(-1)^{p(A)p(B)}BA. \tag{2.5}$$

The standard coproduct map is defined on the generators of SUSY$(\frac{1}{2},\frac{1}{2})$ (2.4)

$$\Delta(g)=g\otimes1+1\otimes g,\qquad \Delta(G)=G\otimes1+1\otimes G.$$

It determines the action of these generators on the tensor product of superfields and yields the standard Leibniz rule for supersymmetry transformations on the local product of superfields $\delta(AB)=(\delta A)B+A\delta B$.

The non-anticommutative deformation $\hat z=(y_m,\hat\theta^\alpha,\bar\theta^{\dot\alpha})$ of the coordinates of the Euclidean $N=(\frac{1}{2},\frac{1}{2})$ superspace was considered in [6]. The basic operator relation of the non-anticommutative superspace is

$$T^{\alpha\beta}(\hat\theta)=\hat\theta^\alpha\star\hat\theta^\beta+\hat\theta^\beta\star\hat\theta^\alpha-C^{\alpha\beta}=0, \tag{2.6}$$

where $C^{\alpha\beta}$ are some constants. The operator superfields $\hat A(y,\hat\theta,\bar\theta)$ and $\hat B(y,\hat\theta,\bar\theta)$ with the antisymmetric ordering of the $\hat\theta^\alpha$ decomposition contain the highest terms $\sim\varepsilon_{\alpha\beta}\hat\theta^\alpha\star\hat\theta^\beta$. In the pseudolocal representation, we consider the usual superfields $A(z)$ and $B(z)$ as the supercommutative images of these operator superfields. The corresponding

$\star$-product of superfields $A(z)$ and $B(z)$ is defined via the generators of the left $N=(\frac{1}{2},0)$ supersymmetry

$$A\star B=Ae^{P}B=AB-\frac{1}{2}(-1)^{p(A)}C^{\alpha\beta}Q_\alpha A\,Q_\beta B-\frac{1}{32}C^{\alpha\beta}C_{\alpha\beta}Q^2A\,Q^2B,$$
$$APB=-\frac{1}{2}(-1)^{p(A)}C^{\alpha\beta}Q_\alpha A\,Q_\beta B,\qquad P^3=0, \tag{2.7}$$

where $p(A)$ is the Z_2 grading, and P is the nilpotent bi-differential operator. The deformed algebras $\mathrm{S}_\star(4|2,2)$ and $\mathrm{C}_\star(4|2,0)$ use this non-commutative product for general or chiral superfields, respectively.

The twist operator in this supersymmetry was introduced in [10] (see also discussions in [11,14])

$$\mathcal{F}=\exp(\mathcal{P}),\quad \mathcal{P}=-\frac{1}{2}C^{\alpha\beta}Q_\alpha\otimes Q_\beta,$$
$$\mathcal{F}(A\otimes B)=A\otimes B-\frac{1}{2}(-1)^{p(A)}C^{\alpha\beta}Q_\alpha A\otimes Q_\beta B-\frac{1}{32}C^{\alpha\beta}C_{\alpha\beta}Q^2A\otimes Q^2B. \tag{2.8}$$

The bilinear map $\mu_\star$ in $\mathrm{S}_\star(4|2,2)$ can be defined via this twist operator

$$A\star B\equiv\mu_\star\circ A\otimes B=\mu\circ\exp(\mathcal{P})A\otimes B, \tag{2.9}$$

so $\mathcal{F}$ is the quantum group analog of the pseudolocal operator e^P (2.7).

By analogy with the map between differential operators on commutative and non-commutative algebras of functions [4], one can easily define the corresponding differential operator $\hat{X}_D$ on $\mathrm{S}_\star(4|2,2)$ for any differential operator D on the supercommutative algebra. In the case of the 1st-order operator $D_1=\xi^M(z)\partial_M$, the image $\hat{X}_{D_1}$ contains, in general, terms with higher derivatives on $\mathrm{S}_\star(4|2,2)$

$$(\hat{\partial}_M\star A)=\partial_M A,\qquad \big(\hat{\partial}_M\star z^N\big)=\delta^N_M,$$
$$(\hat{X}_{D_1}\star A)=(D_1A)=\mu_\star\circ\exp(-\mathcal{P})\big(\xi^M(z)\otimes\partial_M\big)(1\otimes A)$$
$$=\xi^M(z)\star\partial_M A+\frac{1}{2}(-1)^{p(D_1)}C^{\alpha\beta}Q_\alpha\xi^M(z)\star\partial_M Q_\beta A-\frac{1}{32}C^{\alpha\beta}C_{\alpha\beta}Q^2\xi^M(z)\star\partial_M Q^2A, \tag{2.10}$$

where $p(D_1)$ is the Z_2 grading of D_1. For instance, the deformed images of generators $\bar{Q}_{\dot\alpha}$ and L^β_a (2.3) are the following second-order differential operators on $\mathrm{S}_\star(4|2,2)$:

$$(\hat{\bar{Q}}_{\dot\alpha}\star A)=\big(\bar{\partial}_{\dot\alpha}-2i\theta^\alpha\partial_{\alpha\dot\alpha}+iC^{\alpha\beta}\partial_{\alpha\dot\alpha}Q_\beta\big)\star A=\bar{Q}_{\dot\alpha}A,$$
$$\big(\hat{L}^\beta_\alpha\star A\big)=\left(L^\beta_\alpha-\frac{1}{2}C^{\beta\gamma}Q_\gamma Q_\alpha\right)\star A=L^\beta_\alpha A, \tag{2.11}$$

while $\hat{g}=g$ and $\hat{O}=O$. The deformed operators can be used in the operator representation of $\mathrm{S}_\star(4|2,2)$ to check directly the covariance of the basic operator $T^{\alpha\beta}(\hat{\theta})$ (2.6) with respect to the transformations of the deformed supersymmetry

$$\hat{\bar{Q}}_{\dot\alpha}\star T^{\alpha\beta}=0,\qquad \hat{L}^\rho_\sigma\star T^{\alpha\beta}=\delta^\alpha_\sigma T^{\rho\beta}+\delta^\beta_\sigma T^{\alpha\rho}-\delta^\rho_\sigma T^{\alpha\beta}. \tag{2.12}$$

The Lie superalgebra of the deformed generators with hats is isomorphic to the Lie superalgebra of the undeformed supersymmetry generators (2.3).

The coproduct $\Delta_t(G)=e^{-\mathcal{P}}\Delta(G)e^{\mathcal{P}}$ in $\mathrm{SUSY}_t(\frac{1}{2},\frac{1}{2})$ is deformed on $G=\bar{Q}_{\bar\epsilon}+L_\lambda+aO$, in particular,

$$\Delta_t(\bar{Q}_{\bar\epsilon})=\bar{Q}_{\bar\epsilon}\otimes 1+1\otimes\bar{Q}_{\bar\epsilon}+i\bar\epsilon^{\dot\alpha}C^{\alpha\beta}(\partial_{\alpha\dot\alpha}\otimes Q_\beta-Q_\alpha\otimes\partial_{\beta\dot\alpha}),$$
$$\Delta_t(L_\lambda)=L_\lambda\otimes 1+1\otimes L_\lambda+\frac{1}{2}C^{\rho\sigma}\big(\lambda^\alpha_\rho Q_\alpha\otimes Q_\sigma+\lambda^\alpha_\sigma Q_\rho\otimes Q_\alpha\big),$$

$$\Delta_t(O) = O \otimes 1 + 1 \otimes O - C^{\alpha\beta} Q_\alpha \otimes Q_\beta, \tag{2.13}$$

while $e^{-\mathcal{P}}\Delta(g)e^{\mathcal{P}} = \Delta(g) = g \otimes 1 + 1 \otimes g$.

Acting by the composition of $\mu_\star$ and coproduct $\Delta_t(\bar{Q}_{\bar{\epsilon}})$ on the tensor product of superfields, one can obtain the following relation:

$$\begin{aligned}\hat{\delta}_{\bar{\epsilon}} \star (A \star B) &\equiv -\mu_\star \circ \Delta_t(\bar{Q}_{\bar{\epsilon}}) A \otimes B \\ &= -(\bar{Q}_{\bar{\epsilon}} A) \star B - A \star \bar{Q}_{\bar{\epsilon}} B - i\bar{\epsilon}^{\dot{\alpha}} C^{\alpha\beta}\big[(-1)^{p(A)} \partial_{\alpha\dot{\alpha}} A \star Q_\beta B - Q_\alpha A \star \partial_{\beta\dot{\alpha}} B\big] \\ &= -\bar{Q}_{\bar{\epsilon}}(A \star B).\end{aligned} \tag{2.14}$$

The last formula can be derived from the pseudolocal definition of $A \star B$ (2.7)

$$\hat{\delta}_{\bar{\epsilon}} \star (A \star B) = -\big[\bar{Q}_{\bar{\epsilon}}, \big(Ae^P B\big)\big] = (\bar{Q}_{\bar{\epsilon}} A)e^P B + A\big[\bar{Q}_{\bar{\epsilon}}, e^P\big]B + Ae^P(\bar{Q}_{\bar{\epsilon}} B). \tag{2.15}$$

Note that Eq. (2.14) can be treated as the deformed Leibniz rule for $\hat{\delta}_{\bar{\epsilon}}$.

It is important to formulate the principle of covariance of the algebra $\mathrm{S}_\star(4|2,2)$ (or $\mathrm{C}_\star(4|2,0)$) with respect to all deformed transformations of $\mathrm{SUSY}_t(\frac{1}{2},\frac{1}{2})$: the primary superfields A, B and their $\star$-product $(A \star B)$ transform analogously

$$\hat{\delta}_G \star A = -GA = -\hat{G} \star A, \qquad \hat{\delta}_G \star (A \star B) = -G(A \star B) = -\hat{G} \star (A \star B), \tag{2.16}$$

where $\hat{\delta}_G = (\hat{\delta}_{\bar{\epsilon}} + \hat{\delta}_\lambda + \hat{\delta}_a)$. In the pseudolocal formalism of the non-commutative superfield theory, these relations are sufficient to derive the deformed Leibniz rules.

It is evident that the algebra $\mathrm{S}_\star(4|2,2)$ is not covariant with respect to the undeformed supersymmetry, e.g., $\delta_{\bar{\epsilon}}(A \star B) = (\delta_{\bar{\epsilon}} A \star B) + (A \star \delta_{\bar{\epsilon}} B) \neq -\bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}(A \star B)$.

The deformed transformations act non-covariantly on the supercommutative product of superfields AB. For instance, it is not difficult to consider the following transformations of the ordinary product of the even chiral superfields:

$$\begin{aligned}\hat{\delta}_{\bar{\epsilon}} \star (\phi_1\phi_2) &\equiv -\mu \circ \Delta_t(\bar{Q}_{\bar{\epsilon}})\phi_1 \otimes \phi_2 = -\bar{Q}_{\bar{\epsilon}}(\phi_1\phi_2) - i\bar{\epsilon}^{\dot{\alpha}} C^{\alpha\beta}(\partial_{\alpha\dot{\alpha}}\phi_1 Q_\beta\phi_2 - Q_\alpha\phi_1 \partial_{\beta\dot{\alpha}}\phi_2) \\ &= \hat{\delta}_{\bar{\epsilon}}(a_1a_2) + O(\theta), \\ \hat{\delta}_\lambda \star (\phi_1\phi_2) &\equiv -\mu \circ \Delta_t(L_\lambda)\phi_1 \otimes \phi_2 = -\lambda^\alpha_\beta L^\beta_\alpha(\phi_1\phi_2) - \frac{1}{2}C^{\rho\sigma}\big(\lambda^\alpha_\rho Q_\alpha\phi_1 Q_\sigma\phi_2 + \lambda^\alpha_\sigma Q_\rho\phi_1 Q_\alpha\phi_2\big) \\ &= \hat{\delta}_\lambda(a_1a_2) + O(\theta).\end{aligned} \tag{2.17}$$

The first terms coincide with the transformations of the undeformed supersymmetry. Using the θ-decomposition of these superfield formulas one can obtain the deformed transformations of the products of component fields, for instance,

$$\begin{aligned}\hat{\delta}_{\bar{\epsilon}}(a_1a_2) &= -i\bar{\epsilon}^{\dot{\alpha}} C^{\alpha\beta}(\partial_{\alpha\dot{\alpha}}a_1\,\psi_{\beta 2} - \psi_{\alpha 1}\partial_{\beta\dot{\alpha}}a_2), \\ \hat{\delta}_\lambda(a_1a_2) &= -\lambda^\alpha_\beta L^\beta_\alpha(y)(a_1a_2) - \frac{1}{2}C^{\rho\sigma}\big(\lambda^\alpha_\rho\psi_{\alpha 1}\psi_{\sigma 2} + \lambda^\alpha_\sigma\psi_{\rho 1}\psi_{\alpha 2}\big).\end{aligned} \tag{2.18}$$

Let us consider two even chiral superfields ϕ_1 and ϕ_2 in the chiral basis

$$\phi_i = a_i + \theta^\alpha\psi_{i\alpha} + \theta^2 f_i, \qquad Q_\alpha\phi_i = \psi_{i\alpha} + 2\theta_\alpha f_i, \qquad Q^2\phi_i = -4f_i. \tag{2.19}$$

The θ-decomposition of the $\star$-product of two chiral superfields depends on these components and constants $C^{\alpha\beta}$

$$\Phi_{12} = \phi_1 \star \phi_2 = \mathcal{B} + \theta^\alpha\Psi_\alpha + \theta^2 F, \tag{2.20}$$

$$\mathcal{B} = a_1a_2 - \frac{1}{2}C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta} - \frac{1}{2}C^{\alpha\beta}C_{\alpha\beta}f_1f_2, \qquad \Psi_\alpha = a_1\psi_{2\alpha} + a_2\psi_{1\alpha} - C_{\alpha\beta}\big(f_1\psi^\beta_2 - f_2\psi^\beta_1\big),$$

$$F=a_1f_2+a_2f_1-\frac{1}{2}\psi_1^\alpha\psi_{2\alpha}. \tag{2.21}$$

These relations can be treated as a deformed tensor calculus for chiral component multiplets. The t-supersymmetry transformations of the composite components (2.20) are completely analogous to the transformations of the basic components a_i, $\psi_{\alpha i}$, f_i

$$\begin{aligned}&\hat\delta_{\bar\epsilon}\mathcal{B}=0,\qquad \hat\delta_{\bar\epsilon}\Psi_\alpha=-2i\bar\epsilon^{\dot\alpha}\partial_{\alpha\dot\alpha}\mathcal{B},\qquad \hat\delta_{\bar\epsilon}F=-i\bar\epsilon^{\dot\alpha}\partial_{\alpha\dot\alpha}\Psi^\alpha,\\ &\hat\delta_\lambda\mathcal{B}=-\lambda^\alpha_\beta L^\beta_\alpha(y)\mathcal{B},\qquad \hat\delta_\lambda\Psi_\gamma=\lambda^\alpha_\gamma\Psi_\alpha-\lambda^\alpha_\beta L^\beta_\alpha(y)\Psi_\gamma,\qquad \hat\delta_\lambda F=-\lambda^\alpha_\beta L^\beta_\alpha(y)F.\end{aligned} \tag{2.22}$$

These transformations are compatible with the non-covariant component transformations (2.18).

The non-anticommutative deformation of the Euclidean model for an arbitrary number of the chiral and antichiral superfields ϕ_a and $\bar\phi_a$ is based on the superfield action $S_\star(\phi_a,\bar\phi_a)$ [6]. Each term of the $\star$-polynomial decomposition of this action is separately invariant with respect to $\mathrm{SUSY}_t(\frac{1}{2},\frac{1}{2})$, while the quadratic terms like $\int d^8z\,\phi_a\star\bar\phi_a=\int d^8z\,\phi_a\bar\phi_a$ possess also the undeformed supersymmetry.

3. Twist-deformed $N=(1,1)$ supersymmetry

The nilpotent deformations of the Euclidean $N=(1,1)$ supersymmetry were considered in the framework of the harmonic-superspace approach [7,8]. Harmonic-superspace coordinates contain the $\mathrm{SU}(2)/\mathrm{U}(1)$ harmonics $u^\pm_i$ and the chiral superspace coordinates

$$z^M=\left(y_m,\theta^\alpha_k,\bar\theta^{\dot\alpha k}\right),\quad y_m=x_m+i\theta_k\sigma_m\bar\theta^k, \tag{3.1}$$

where x_m are the central 4D coordinates. The spinor derivatives D^k_α and $\bar D_{\dot\alpha k}$ in these coordinates are

$$D^k_\alpha=\partial^k_\alpha+2i\bar\theta^{\dot\alpha k}\partial_{\alpha\dot\alpha},\qquad \bar D_{\dot\alpha k}=\bar\partial_{\dot\alpha k}. \tag{3.2}$$

Using harmonic projections of the Grassmann coordinates $\theta^{\pm\alpha}=u^\pm_k\theta^{\alpha k}$ and $\bar\theta^{\pm\dot\alpha}=u^\pm_k\bar\theta^{\dot\alpha k}$ one can define the analytic coordinates $(x_A,\theta^\pm,\bar\theta^\pm)$. The corresponding representation of spinor derivatives and supersymmetry generators can be found in [7,12]. The bosonic part of the harmonic superspaces is $R^4\times S^2$, and the left–right Grassmann dimensions of general, chiral or analytic superspaces are $(4,4)$, $(4,0)$ or $(2,2)$, respectively. We shall use the notation $\mathrm{S}(4,2|4,4)$ for the supercommutative algebra and $\mathrm{S}_\star(4,2|4,4)$ for the non-anticommutative algebra of general harmonic superfields, respectively.

It is convenient to use the following differential representation of the $\mathrm{SUSY}(1,1)$ generators on $\mathrm{S}(4,2|4,4)$:

$$\begin{aligned}&T^k_l=-\theta^\alpha_l\partial^k_\alpha+\frac{1}{2}\delta^k_l\theta^\alpha_j\partial^j_\alpha+\bar\theta^{\dot\alpha k}\bar\partial_{\dot\alpha l}-\frac{1}{2}\delta^k_l\bar\theta^{\dot\alpha j}\bar\partial_{\dot\alpha j}-u^\pm_l\partial^{\mp k}+\frac{1}{2}\delta^k_l u^\pm_j\partial^{\mp j},\\ &L^\beta_\alpha=L^\beta_\alpha(y)+\theta^\beta_k\partial^k_\alpha-\frac{1}{2}\delta^\beta_\alpha\theta^\gamma_k\partial^k_\gamma,\qquad R^{\dot\beta}_{\dot\alpha}=R^{\dot\beta}_{\dot\alpha}(y)+\bar\theta^{\dot\beta k}\bar\partial_{\dot\alpha k}-\frac{1}{2}\delta^{\dot\beta}_{\dot\alpha}\bar\theta^{\dot\gamma k}\bar\partial_{\dot\gamma k},\\ &O=\theta^\alpha_k\partial^k_\alpha-\bar\theta^{\dot\alpha k}\bar\partial_{\dot\alpha k},\qquad Q^k_\alpha=\partial^k_\alpha,\qquad \bar Q_{\dot\alpha k}=\bar\partial_{\dot\alpha k}-2i\theta^\alpha_k\partial_{\alpha\dot\alpha},\qquad P_m=\partial_m\end{aligned} \tag{3.3}$$

where $L^\beta_\alpha(y)$ and $R^{\dot\beta}_{\dot\alpha}(y)$ are defined above (2.3), and partial harmonic derivatives act as follows $\partial^{\mp l}u^\pm_k=\delta^l_k$. For our purposes, it is convenient to consider the following combinations of SUSY(1,1) generators and corresponding parameters:

$$\begin{aligned}&g=P_c+R_\rho+Q_\epsilon,\qquad G=T_u+L_\lambda+\bar Q_{\bar\epsilon}+aO,\\ &\quad P_c=c_mP_m,\quad T_u=u^k_lT^l_k,\quad L_\lambda=\lambda^\alpha_\beta L^\beta_\alpha,\quad R_\rho=\rho^{\dot\alpha}_{\dot\beta}R^{\dot\beta}_{\dot\alpha},\quad Q_\epsilon=\epsilon^\alpha_kQ^k_\alpha,\quad \bar Q_{\bar\epsilon}=\bar\epsilon^{\dot\alpha k}\bar Q_{\dot\alpha k}.\end{aligned} \tag{3.4}$$

The $N=(1,1)$ twist operator $\mathcal{F}=\exp(\mathcal{P})$ contains the nilpotent operator

$$\mathcal{P}=-\frac{1}{2}C^{\alpha\beta}_{kl}Q^k_\alpha\otimes Q^l_\beta,\qquad \mathcal{P}^5=0, \tag{3.5}$$

where $C^{\alpha\beta}_{kl}$ are some constants. The non-anticommutative product in the corresponding deformed algebra $\mathrm{S}_\star(4,2|4,4)$ can be defined by equivalent formulas

$$A\star B=A\exp(P)B=\mu\circ\exp(\mathcal{P})A\otimes B=\mu_\star\circ A\otimes B, \tag{3.6}$$

where μ and $\mu_\star$ are product maps for $\mathrm{S}(4,2|4,4)$ and $\mathrm{S}_\star(4,2|4,4)$ and P is the basic operator from [7,8]

$$APB=-\frac{1}{2}(-1)^{p(A)}C^{\alpha\beta}_{kl}Q^k_\alpha A\,Q^l_\beta B=\mu\circ\mathcal{P}\,A\otimes B. \tag{3.7}$$

The non-anticommutative algebras of the $N=(1,1)$ chiral or G-analytic superfields can be defined analogously.

By analogy with Eq. (2.10), one can construct the image operator $\hat{X}_{D_1}$ on $\mathrm{S}_\star(4,2|4,4)$ for any 1st-order differential operator $D_1=\xi^M(z)\partial_M$ on $\mathrm{S}(4,2|4,4)$. The twisted coproduct of $\mathrm{SUSY}_t(1,1)\Delta_t(G)=e^{-\mathcal{P}}\Delta(G)e^{\mathcal{P}}$ is deformed on $G=(\bar{Q}_{\bar\epsilon}+T_u+L_\lambda+aO)$, in particular,

$$\begin{aligned}
&\Delta_t(\bar{Q}_{\bar\epsilon})=\bar{Q}_{\bar\epsilon}\otimes 1+1\otimes\bar{Q}_{\bar\epsilon}+i\bar\epsilon^{\dot\alpha k}C^{\alpha\beta}_{kj}\partial_{\alpha\dot\alpha}\otimes Q^j_\beta-i\bar\epsilon^{\dot\alpha k}C^{\alpha\beta}_{ik}Q^i_\alpha\otimes\partial_{\beta\dot\alpha},\\
&\Delta_t(T_u)=T_u\otimes 1+1\otimes T_u-\frac{1}{2}u^l_kC^{\alpha\beta}_{lj}Q^k_\alpha\otimes Q^j_\beta-\frac{1}{2}u^l_kC^{\alpha\beta}_{jl}Q^j_\alpha\otimes Q^k_\beta,\\
&\Delta_t(L_\lambda)=L_\lambda\otimes 1+1\otimes L_\lambda+\frac{1}{2}\lambda^\alpha_\beta C^{\beta\gamma}_{kl}Q^k_\alpha\otimes Q^l_\gamma+\frac{1}{2}\lambda^\alpha_\beta C^{\rho\beta}_{kl}Q^k_\rho\otimes Q^l_\alpha,\\
&\Delta_t(O)=O\otimes 1+1\otimes O-C^{\alpha\beta}_{kl}Q^k_\alpha\otimes Q^l_\beta,
\end{aligned} \tag{3.8}$$

while $e^{-\mathcal{P}}\Delta(g)e^{\mathcal{P}}=\Delta(g)$ for $g=Q_\epsilon+P_c+R_\rho$.

The $\star$-products of arbitrary $N=(1,1)$ superfields preserve covariance with respect to all deformed transformations of $\mathrm{SUSY}_t(1,1)$

$$\hat\delta_G\star(A\star B)=-\mu_\star\circ\Delta_t(G)A\otimes B=-G(A\star B). \tag{3.9}$$

The deformed Leibniz rules can be derived directly from these covariant relations.

The twisted supersymmetry acts non-covariantly on the supercommutative product of superfields, for instance,

$$\begin{aligned}
\hat\delta_{\bar\epsilon}\star(AB)&\equiv-\mu\circ\Delta_t(\bar{Q}_{\bar\epsilon})A\otimes B\\
&=-\bar{Q}_{\bar\epsilon}(AB)-i\bar\epsilon^{\dot\alpha k}C^{\alpha\beta}_{kj}(-1)^{p(A)}\partial_{\alpha\dot\alpha}A\,Q^j_\beta B+i\bar\epsilon^{\dot\alpha k}C^{\alpha\beta}_{ik}Q^i_\alpha A\,\partial_{\beta\dot\alpha}B.
\end{aligned} \tag{3.10}$$

It is easy to define the t-deformed transformations on the products of the $N=(1,1)$ component fields using the corresponding Grassmann decompositions.

In the special case of the singlet deformation [12,13], the twist operator is defined by the parameter I and the $\mathrm{SU}(2)\times\mathrm{SU}(2)_L$ invariant constant tensor

$$C^{\alpha\beta}_{kl}=2I\varepsilon^{\alpha\beta}\varepsilon_{kl}\ \Rightarrow\ \mathcal{P}_s=-I\,Q^i_\alpha\otimes Q^\alpha_i. \tag{3.11}$$

The $\mathcal{P}_s$-twist deformation vanishes for $\mathrm{SU}(2)$ and $\mathrm{SU}(2)_L$ transformations.

The Leibniz rules for differential operators $D=(\partial_m,D^k_\alpha,\bar{D}_{\dot\alpha k},\partial/\partial u^\pm_k)$ are standard for the general Q-deformation (3.5)

$$D(A\star B)=(DA\star B)+(-1)^{p(D)p(A)}(A\star DB). \tag{3.12}$$

The $\star$-product preserves differential constraints of chirality, antichirality and Grassmann analyticity [7,8]. All superfield actions using $\star$-products in the non-anticommutative $N=(1,1)$ harmonic superspace [12,13] are invariant with respect to the quantum group $\mathrm{SUSY}_t(1,1)$, and this invariance is a natural basic principle of these deformed theories. Free quadratic parts of these actions possess also the undeformed $N=(1,1)$ supersymmetry. The simple examples of the superfield-density terms for the analytic hypermultiplet $q^+, \tilde{q}^+$ and the U(1) gauge potential V^{++} in the deformed theory are

$$\tilde{q}^+ \star \left(D^{++}q^+ + \left[V^{++}, q^+\right]_\star\right) + \lambda q^+ \star q^+ \star \tilde{q}^+ \star \tilde{q}^+. \tag{3.13}$$

The t-supersymmetry transformations of any term $L^{+4}_\star$ in this density are

$$\hat{\delta}_G \star L^{+4}_\star = -\left(\bar{\epsilon}^{\dot{\alpha}k} \bar{Q}_{\dot{\alpha}k} + u^l_k T^k_l + l^\alpha_\beta L^\beta_a + a O\right) L^{+4}_\star, \tag{3.14}$$

so the analytic-superspace integrals of these variations vanish. We hope that the manifest $\mathrm{SUSY}_t(1,1)$ covariance could help to prove the non-renormalization theorems in t-deformed harmonic-superfield theories by analogy with the corresponding undeformed theories.

4. Conclusions

We analyzed the twist deformations of the Euclidean $N=(\frac{1}{2},\frac{1}{2})$ and $N=(1,1)$ supersymmetries. By analogy with the formalism of the deformed Minkowski space [4], we construct explicitly the map between differential operators on ordinary and deformed superspaces (2.10). This map connects the standard representation for the supersymmetry generators with the corresponding operator representation on the deformed superspace. It is shown that the non-commutative $\star$-products of primary superfields transform covariantly in these t-deformed supersymmetries. This covariance is a basic principle of the superfield formalism of the deformed theories. The Grassmann-coordinate decompositions of the $\star$-product superfields define the deformed tensor calculus for the components of primary superfields. The ordinary supercommutative products of primary superfields or component fields are not covariant with respect to the deformed supersymmetries.

Any polynomial terms of the superfield actions in the non-anticommutative $N=(\frac{1}{2},\frac{1}{2})$ [6] and $N=(1,1)$ [7,8] superspaces are manifestly invariant with respect to the corresponding t-deformed supersymmetries. The bilinear free parts of these actions are also invariant under the standard supersymmetry transformations. The deformation constants of the non-anticommutative superfield theories break some undeformed (super)symmetries, however, these parameters can be treated as ‘coupling constants’ compatible with the deformed supersymmetries. We hope that t-supersymmetries would help to analyze non-renormalization theorems using the superfield effective actions in these theories.

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